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Semilinear elliptic equations in symmetric domains

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Abstract

In this note, we review the author's recent result in [12] on the existence of asymmetric positive solutions for semilinear elliptic equations in symmetric domains.

1 Introduction

We prove the existence of positive solutions without symmetry for the generalized Hénon equation in symmetric domains

$$-\Delta u = h(x)u^p, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.1)$$

Here Ω is a bounded domain in \mathbb{R}^N with piecewise smooth boundary. We assume that $1 < p < \infty$ for $N = 2$, $1 < p < (N + 2)/(N - 2)$ for $N \geq 3$, $h \in L^\infty(\Omega)$ and $h(x)$ may or may not change its sign. Let G be a closed subgroup of the orthogonal group $O(N)$ such that $G \neq \{I\}$, where I is the unit matrix. We call Ω a *G invariant domain* if $g(\Omega) = \Omega$ for any $g \in G$ and $h(x)$ a *G invariant function* if $h(gx) = h(x)$ for any $g \in G$ and $x \in \Omega$. In the same way, a *G invariant solution* is defined. Assume that $h_+(x) := \max(h(x), 0) \not\equiv 0$ in Ω . Then (1.1) has a *G invariant positive solution*. However, we are looking for a solution without *G invariance*. To this end, we define the Rayleigh quotient $R(u)$ with the definition domain $D(R)$ by

$$R(u) := \left(\int_{\Omega} |\nabla u|^2 dx \right) / \left(\int_{\Omega} h(x) |u|^{p+1} dx \right)^{2/(p+1)},$$

$$D(R) := \{u \in H_0^1(\Omega) : \int_{\Omega} h(x)|u|^{p+1}dx > 0\}.$$

Moreover, we define the Nehari manifold \mathcal{N} by

$$\mathcal{N} := \{u \in H_0^1(\Omega) \setminus \{0\} : \int_{\Omega} (|\nabla u|^2 - h(x)|u|^{p+1})dx = 0\}.$$

The *least energy* R_0 is defined by

$$R_0 := \inf\{R(u) : u \in D(R)\} = \inf\{R(u) : u \in \mathcal{N}\}. \quad (1.2)$$

We call u a *least energy solution* if $u \in \mathcal{N}$ and $R(u) = R_0$. It becomes a positive or negative solution of (1.1). We choose a positive one as a least energy solution after replacing u by $-u$, if necessary.

To explain our purpose, we introduce the Hénon equation

$$-\Delta u = |x|^{\lambda}u^p, \quad u > 0 \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B, \quad (1.3)$$

where B is the unit ball in \mathbb{R}^N . Smets, Willem and Su [17] have proved that if λ is large enough, then a least energy solution of (1.3) is not radially symmetric. It is known that there exists a radial positive solution. Therefore (1.3) has both a radial positive solution and a nonradial positive solution. There are many papers which have studied the Hénon equation ([1, 2, 3, 4, 5, 6, 7, 8, 9, 15, 16]).

On the other hand, Moore and Nehari [13, pp.32–33] have studied the two point boundary value problem of the ordinary differential equation

$$u''(t) + h(t)u^p = 0, \quad u > 0 \quad \text{in } (-1, 1), \quad u(-1) = u(1) = 0, \quad (1.4)$$

where $h(t) = 0$ for $|t| < a$ and $h(t) = 1$ for $a < |t| < 1$. When $a(< 1)$ is sufficiently close to 1, they have constructed at least three positive solutions of (1.4): an even solution $u(t)$, a non-even solution $v(t)$ and the reflection $v(-t)$. The purpose of this paper is to extend the results above to more general symmetric domains Ω and to more general weight functions $h(x)$.

2 Main result

In this section, we state main results and give several examples of Ω and $h(x)$. We first define the *fixed point set* of G by

$$F = \text{Fix}(G) := \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in G\}.$$

Then F is a linear subspace of \mathbb{R}^N . Since $G \neq \{I\}$ is assumed with the unit matrix I , it holds that $0 \leq \dim F \leq N - 1$.

Definition 2.1. Let F^\perp be the orthogonal complement of F in \mathbb{R}^N . We denote by $x = x' + x''$ the orthogonal decomposition of x into $x' \in F$ and $x'' \in F^\perp$. We define

$$\text{dist}(x, F) := \inf\{|x - y| : y \in F\} = |x''|,$$

$$\Omega(|x''| < a) := \{x' + x'' \in \Omega : |x''| < a\} \quad \text{for } a > 0.$$

Put

$$L := \max\{\text{dist}(x, F) : x \in \overline{\Omega}\} = \max\{|x''| : x' + x'' \in \overline{\Omega}\}.$$

We denote the set of the farthest points in $\overline{\Omega}$ from F by $\partial\Omega_0$, i.e.,

$$\partial\Omega_0 := \{x \in \partial\Omega : \text{dist}(x, F) = L\}.$$

Assumption 2.2. Assume that $h(x)$ satisfies either (A) or (B) below.

(A) Let $h(x)$ take the form $h(x) = f(x)^\lambda$ with $\lambda > 0$ large enough, where $f(x)$ is a G invariant continuous function on $\overline{\Omega}$ such that

$$0 \leq f(x) < \max_{y \in \partial\Omega_0} f(y) \quad \text{for } x \in \overline{\Omega} \setminus \partial\Omega_0.$$

(B) $h(x) \leq 0$ in $\Omega(|x''| < a)$, $h_+(x) \not\equiv 0$ in $\Omega(a < |x''| < L)$ and $a \in (0, L)$ is sufficiently close to L .

We state our main result in the following.

Theorem 2.3. *Let Ω and h be G invariant and h satisfy either (A) or (B). Then a least energy solution of (1.1) is not G invariant. Therefore (1.1) has both a G invariant positive solution and a G non-invariant positive solution.*

When $\text{Fix}(G) = \{0\}$ and $h(r)$ is radial, conditions (A) and (B) reduce to the following conditions.

(A)' $h(r) = f(r)^\lambda$ with λ large enough and $0 \leq f(r) < f(L)$ for $0 \leq r < L$.

(B)' $h(r) \leq 0$ in $(0, a)$, $h_+(r) \not\equiv 0$ in (a, L) and a is sufficiently close to L .

Examples of $h(x)$ satisfying (A)' are $h(|x|) = |x|^\lambda$, $e^{\lambda|x|}$, $(|x|/(1+|x|))^\lambda$. A simple example of h satisfying (B)' is $h(|x|) = (|x| - a)/(L - a)$.

Corollary 2.4. *Suppose that $\text{Fix}(G) = \{0\}$ and $h(r)$ satisfies either (A)' or (B)'. Then the same conclusion as in Theorem 2.3 holds.*

Example 2.5. Let $G = O(N)$ and Ω be a ball with radius L . Then $\text{Fix}(G) = \{0\}$. Let $h(r)$ satisfy either (A)' or (B)'. Then a least energy solution is not radially symmetric. This example extends the result by Smets, Willem and Su [17] to more general $h(x)$.

Example 2.6. Let Ω be a convex regular polytope with center origin in \mathbb{R}^N . We define the regular polytope group $G(\Omega)$ by

$$G(\Omega) := \{g \in SO(N) : g(\Omega) = \Omega\},$$

where $SO(N)$ denotes the rotation group. Then it holds that $\text{Fix}(G(\Omega)) = \{0\}$ for any regular polytope Ω . Let $h(r)$ satisfy either (A)' or (B)', where L is the radius of a circumscribed sphere of Ω . Then a least energy solution of (1.1) is not invariant under the action of the regular polytope group $G(\Omega)$.

Example 2.7. Let Ω be a cylinder in \mathbb{R}^3 , which is defined by

$$\Omega := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < \alpha^2, |x_3| < \beta\},$$

with $\alpha, \beta > 0$. Put $L := \sqrt{\alpha^2 + \beta^2}$ and let $h(r)$ be a radially symmetric function satisfying (A)' or (B)'. Then a least energy solution is not even, not rotationally symmetric around the x_3 -axis and not reflectionally symmetric with respect to the plane $x_3 = 0$.

We shall prove this assertion. First, we choose $G := \{I, -I\}$. Then $\text{Fix}(G) = \{0\}$. By Corollary 2.4, a least energy solution is not even.

Next, we choose

$$G := \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} : g \in O(2) \right\}.$$

Then G invariance means the rotational invariance around the x_3 -axis. By Theorem 2.3, a least energy solution is not rotationally invariant around the x_3 -axis.

Lastly, we choose

$$G = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} I_2 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

where I_2 denotes the 2×2 unit matrix. By Theorem 2.3, a least energy solution is not reflectionally symmetric with respect to the plane $x_3 = 0$.

Example 2.8. Let

$$\Omega := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0, (1 \leq i \leq 3), x_1 + x_2 + x_3 < 1\}.$$

Then Ω is a triangular pyramid. Let $h(r)$ satisfy either (A)' or (B)' with $L = 1$. Then a least energy solution is not invariant under the rotation by angle $2\pi/3$ around the line $x_1 = x_2 = x_3$.

3 Proof of the main theorem

We give a sketch of proof of Theorem 2.3. The next lemma is known, but we give a proof for the reader's convenience.

Lemma 3.1. *Let u be a positive solution of (1.1). Then we have*

$$0 < \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} hu^{p+1} dx, \quad (3.1)$$

$$R(u) = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{(p-1)/(p+1)} = \left(\int_{\Omega} hu^{p+1} dx \right)^{(p-1)/(p+1)}. \quad (3.2)$$

Proof. Multiplying (1.1) by u and integrating it over Ω , we obtain (3.1), which leads to (3.2). \square

To prove Theorem 2.3, we define

$$H_0^1(\Omega, G) := \{u \in H_0^1(\Omega) : u \text{ is } G \text{ invariant}\},$$

$$D(R, G) := D(R) \cap H_0^1(\Omega, G), \quad \mathcal{N}(G) := \mathcal{N} \cap H_0^1(\Omega, G).$$

We define a G invariant least energy

$$R_G := \inf\{R(u) : u \in D(R, G)\} = \inf\{R(u) : u \in \mathcal{N}(G)\}.$$

We call u a G invariant least energy solution if $u \in \mathcal{N}(G)$ and $R(u) = R_G$. We call R_0 a global least energy, which has already been defined by (1.2). To prove the theorem, it is enough to show that $R_0 < R_G$. Indeed, this inequality ensures that a global least energy solution corresponding to R_0 cannot be G invariant because R_G is the infimum of $R(u)$ for all G invariant solutions u .

Let us show $R_0 < R_G$. Let u be a G invariant least energy solution. We shall define $\phi(x)$ later on, which satisfies

$$R((1 + \varepsilon\phi)u) < R(u) = R_G \quad \text{for } \varepsilon > 0 \text{ small enough.} \quad (3.3)$$

Putting $v(x) := (1 + \varepsilon\phi)u$, we obtain

$$R_0 \leq R(v) < R(u) = R_G.$$

We shall construct a function $\phi(x)$ satisfying (3.3). For simplicity of discussion, we consider the case where $N = 2$ and Ω is a regular triangle in \mathbb{R}^2 and G is given by

$$G := \{g(2j\pi/3) : j = 0, 1, 2\}, \quad g(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (3.4)$$

Let $B(x, r)$ denote the ball centered at x with radius r . Let x_0 and x_1 be vertices of the regular triangle Ω such that $g(2\pi/3)(x_0) = x_1$, where $g(2\pi/3)$ is defined by (3.4). We take two small balls $B_0 = B(x_0, 2r_0)$ and $B_1 = B(x_1, 2r_0)$ with the same radius $2r_0$ small enough. Therefore $g(2\pi/3)(B_0) = B_1$. Let $\phi_0 \in C_0^\infty(\mathbb{R}^2)$ be a radially symmetric function such that $0 \leq \phi_0(x) \leq 1$ in \mathbb{R}^2 and

$$\phi_0(x) = 1 \quad \text{for } |x| < r_0, \quad \text{supp}\phi_0 \subset B(0, 2r_0).$$

Here $\text{supp}\phi_0$ denotes the support of ϕ_0 .

Definition 3.2. We define $\phi(x)$ in the whole space \mathbb{R}^2 by

$$\phi(x) := \begin{cases} \phi_0(x - x_0) & \text{if } x \in B(x_0, 2r_0), \\ -\phi_0(x - x_1) & \text{if } x \in B(x_1, 2r_0), \\ 0 & \text{otherwise.} \end{cases}$$

We define the inner product in $H_0^1(\Omega)$ by

$$(u, v)_{H_0^1} := \int_{\Omega} \nabla u \nabla v dx.$$

The orthogonal complement of $H_0^1(\Omega, G)$ in $H_0^1(\Omega)$ is denoted by $H_0^1(\Omega, G)^\perp$, i.e.,

$$H_0^1(\Omega, G)^\perp := \{u \in H_0^1(\Omega) : (u, v)_{H_0^1} = 0 \quad \text{for all } v \in H_0^1(\Omega, G)\}.$$

Then $\phi(x)$ defined above satisfies

$$\phi \in C_0^\infty(\Omega_1) \cap H_0^1(\Omega_1, G)^\perp \cap L^2(\Omega_1, G)^\perp, \quad (3.5)$$

where Ω_1 is defined by

$$\Omega_1 := \{x \in \mathbb{R}^2 : \text{dist}(x, \Omega) < 1\}.$$

Proposition 3.3 ([12]). *Let u be a G invariant least energy solution and ϕ be defined by Definition 3.2. Then we have*

$$\int_{\Omega} |\nabla \phi|^2 u^2 dx - 2(p-1) \int_{\Omega} u \phi \nabla u \nabla \phi dx < (p-1) \int_{\Omega} |\nabla u|^2 \phi^2 dx. \quad (3.6)$$

Using the proposition above, we prove Theorem 2.3 in the following.

Proof of Theorem 2.3. We compute $R(v)$ for $v := (1 + \varepsilon\phi)u$. Multiplying (1.1) by $\phi^2 u$ and integrating it over Ω , we have

$$\int_{\Omega} (|\nabla u|^2 \phi^2 + 2u\phi \nabla u \nabla \phi) dx = \int_{\Omega} hu^{p+1} \phi^2 dx. \quad (3.7)$$

Combining (3.6) with (3.7), we have

$$\frac{1}{p-1} \int_{\Omega} |\nabla \phi|^2 u^2 dx < \int_{\Omega} (|\nabla u|^2 \phi^2 + 2u\phi \nabla u \nabla \phi) dx = \int_{\Omega} hu^{p+1} \phi^2 dx.$$

Hence

$$\int_{\Omega} hu^{p+1} \phi^2 dx > 0. \quad (3.8)$$

Using (3.6) and (3.7), we obtain

$$\begin{aligned} & \int_{\Omega} (|\nabla u|^2 \phi^2 + 2u\phi \nabla u \nabla \phi + |\nabla \phi|^2 u^2) dx \\ & < p \int_{\Omega} (|\nabla u|^2 \phi^2 + 2u\phi \nabla u \nabla \phi) dx = p \int_{\Omega} hu^{p+1} \phi^2 dx. \end{aligned}$$

Choose $q \in (1, p)$ slightly less than p such that

$$\int_{\Omega} (|\nabla u|^2 \phi^2 + 2u\phi \nabla u \nabla \phi + |\nabla \phi|^2 u^2) dx < q \int_{\Omega} hu^{p+1} \phi^2 dx. \quad (3.9)$$

We expand $|\nabla v|^2$ as

$$\begin{aligned} |\nabla v|^2 &= (1 + 2\varepsilon\phi + \varepsilon^2\phi^2)|\nabla u|^2 + 2\varepsilon u \nabla u \nabla \phi \\ &\quad + 2\varepsilon^2 u \phi \nabla u \nabla \phi + \varepsilon^2 |\nabla \phi|^2 u^2. \end{aligned} \quad (3.10)$$

From now on, we extend u onto the whole space \mathbb{R}^2 by putting $u = 0$ outside of Ω . Then $u \in H^1(\Omega_1)$. By (3.5), ϕ and $|\nabla u|^2$ are orthogonal in $L^2(\Omega_1)$, i.e., the integral of $\phi|\nabla u|^2$ on Ω_1 , or equivalently on Ω , is zero. By the same reason, the integral of $2u\nabla u \nabla \phi$ on Ω vanishes. Integrating both sides of (3.10) over Ω , we get

$$\begin{aligned} \int_{\Omega} |\nabla v|^2 dx &= \int_{\Omega} (1 + \varepsilon^2\phi^2)|\nabla u|^2 dx + 2\varepsilon^2 \int_{\Omega} u\phi \nabla u \nabla \phi dx \\ &\quad + \varepsilon^2 \int_{\Omega} |\nabla \phi|^2 u^2 dx \\ &< \int_{\Omega} |\nabla u|^2 dx + \varepsilon^2 q \int_{\Omega} hu^{p+1} \phi^2 dx, \end{aligned}$$

where we have used (3.9). Observing Lemma 3.1, we put

$$A := \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} h u^{p+1} dx, \quad B := \frac{1}{A} \int_{\Omega} h u^{p+1} \phi^2 dx.$$

Then

$$\int_{\Omega} |\nabla v|^2 dx < A(1 + \varepsilon^2 q B). \quad (3.11)$$

Next, we shall compute the denominator of $R(v)$. Expanding $(1 + \varepsilon\phi)^{p+1}$ by the Taylor theorem, we get

$$\begin{aligned} \int_{\Omega} h(x) v^{p+1} dx &= \int_{\Omega} h(x) (1 + \varepsilon\phi)^{p+1} u^{p+1} dx \\ &= \int_{\Omega} h(x) u^{p+1} dx + \varepsilon(p+1) \int_{\Omega} h(x) \phi u^{p+1} dx \\ &\quad + \frac{\varepsilon^2 p(p+1)}{2} \int_{\Omega} h(x) u^{p+1} \phi^2 (1 + \psi) dx. \end{aligned}$$

Here $\psi(x, \varepsilon)$ in the last integral is a remainder term, which converges to zero uniformly on $\bar{\Omega}_1$ as $\varepsilon \rightarrow 0$. Since ϕ is orthogonal to any G invariant function, we have

$$\int_{\Omega} h \phi u^{p+1} dx = 0.$$

Therefore

$$\begin{aligned} \int_{\Omega} h(x) v^{p+1} dx &= \int_{\Omega} h(x) u^{p+1} dx \\ &\quad + \frac{\varepsilon^2 p(p+1)}{2} \int_{\Omega} h(x) u^{p+1} \phi^2 (1 + \psi) dx \\ &= A(1 + \varepsilon^2 C_{\varepsilon}), \end{aligned}$$

where

$$C_{\varepsilon} := \frac{p(p+1)}{2A} \int_{\Omega} h(x) u^{p+1} \phi^2 (1 + \psi(x, \varepsilon)) dx.$$

Observe the easy inequality

$$(1+t)^{-q} \leq 1 - qt(1+t)^{-q-1} \quad \text{for } t \geq 0, \quad q > 0.$$

Substituting $t = \varepsilon^2 C_{\varepsilon}$ and $q = 2/(p+1)$, we have

$$\begin{aligned} &\left(\int_{\Omega} h(x) v^{p+1} dx \right)^{-2/(p+1)} \\ &= A^{-2/(p+1)} (1 + \varepsilon^2 C_{\varepsilon})^{-2/(p+1)} \\ &\leq A^{-2/(p+1)} \left\{ 1 - \frac{2\varepsilon^2}{p+1} (1 + \varepsilon^2 C_{\varepsilon})^{-(p+3)/(p+1)} C_{\varepsilon} \right\} \\ &= A^{-2/(p+1)} (1 - \varepsilon^2 p \theta_{\varepsilon} D_{\varepsilon}), \end{aligned} \quad (3.12)$$

where we have put

$$\theta_\varepsilon := (1 + \varepsilon^2 C_\varepsilon)^{-(p+3)/(p+1)},$$

$$D_\varepsilon := \frac{1}{A} \int_{\Omega} h u^{p+1} \phi^2 (1 + \psi(x, \varepsilon)) dx.$$

Combining (3.11) with (3.12), we get

$$\begin{aligned} R(v) &= \left(\int_{\Omega} |\nabla v|^2 dx \right) \left(\int_{\Omega} h v^{p+1} dx \right)^{-2/(p+1)} \\ &< A^{(p-1)/(p+1)} \{1 + \varepsilon^2 (qB - p\theta_\varepsilon D_\varepsilon)\}. \end{aligned}$$

By (3.8), we have

$$\lim_{\varepsilon \rightarrow 0} p\theta_\varepsilon D_\varepsilon = \frac{p}{A} \int_{\Omega} h u^{p+1} \phi^2 dx > 0.$$

Since $q < p$, it holds that

$$qB = \frac{q}{A} \int_{\Omega} h u^{p+1} \phi^2 dx < \frac{p}{A} \int_{\Omega} h u^{p+1} \phi^2 dx.$$

Therefore $qB < p\theta_\varepsilon D_\varepsilon$ for $\varepsilon > 0$ small enough. Then we have $R(v) < A^{(p-1)/(p+1)} = R(u)$, where the last equation follows from Lemma 3.1. The proof is complete. \square

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